As remarked before, we basically learned everything about Lagrange multipliers.

Problem: Global max/min may not exist if domain is not compact

Recall : Compact = Closed AND Bounded

$\Rightarrow$ Not Compact = Not Closed OR Not Bounded.

$\Rightarrow$ { Being not closed causes a problem,
Being not bounded causes a problem too.

You can already see why in a single variable case.

Ex (Being not closed $\Rightarrow$ Global max/min may not exist).

Consider: $f(x) = x$ on domain $\boxed{\{-1 < x < 1\}}$

Bounded, but NOT CLOSED.

Say you pick any number a in the domain.

In other words, pick any $-1 < a < 1$.

$\Rightarrow$ Can always find another number b in between a & 1,

$-1 < a < b < 1$. $\Rightarrow$ $f(a) = a < f(b) = b$.

$\Rightarrow$ NO GLOBAL MAX. (No point in the domain has the largest value.)

The problem is $f(x)$ wants to have 1 as global max but this endpoint is not included in the domain, cannot be reached.

Similar logic ⇨ NO GLOBAL MIN b/c the left endpoint is not included in the domain.

Ex (Being not bounded ⇨ Global max/min may not exist).

Consider: $f(x)=x$ on domain $\boxed{\{\text{all numbers } x\}}$

Closed, but NOT BOUNDED.

Say you pick any number a. ⇨ Can always find another number b bigger than a, $a<b$.

⇨ $f(a)=a<f(b)=b$ ⇨ NO GLOBAL MAX

(No point in the domain has the largest value.)

The problem is that there is no upper limit in choosing a number in the domain.

Similar logic ⇨ NO GLOBAL MIN b/c there is no lower limit in choosing a number in the domain.

In this course we will only focus on treating the problem coming from an unbounded (= not bounded) domain.

All global max/min problems in this course will have a closed domain.

Reason: In real-world applications, you rarely see a problem with a non-closed domain.

The global max/min may not exist, but actually this is the only possible new outcome.

Theorem Say you have the following problem.

$\boxed{\text{Find a global max/min value of a function } f \text{ on a closed domain.}}$

The following is the procedure of finding global max/min values we learned so far (it works correctly when domain is compact)

> You find (Lagrange) critical points of the original domain and its boundary pieces, according to one of the tables in the previous lecture note.
>
> You get a list of points.
>
> You compute the list of values of the function on those points on the list of points.
>
> You take the largest value L
> the smallest value S

If the domain is closed but not bounded, then

- 2 possibilities for global max value:

  EITHER
  - global max value $= L$ (largest value from the above procedure)

  OR
  - global max does not exist.

- 2 possibilities for global min value:

  EITHER
  - global min value $= S$ (smallest value from the above procedure)

  OR
  - global min does not exist.

There is no general DNE checking process, similar to that there is no general way of solving limit problem.

We will content with one case that guarantees that the global minimum exists, even on the unbounded domains.

Global minimum of coercive functions

Global minimum always exists on any closed domain for coercive functions.

Definition (Coercive functions)

A function $f(x,y)$ is coercive if the domain $\{f(x,y) \le k\}$ is bounded for any number k.

Empty domains are considered bounded.

Example $f(x,y) = x^2+y^2$ is coercive.

Because we know any domain of the form $\{x^2+y^2 \le k\}$ is bounded.

Similar definition for 3 variable functions.

Example $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ is coercive.

Similarly, any domain of the form

$\{\sqrt{x^2+y^2+z^2} \le k\}$ is the inside of a sphere, bounded.

Example $f(x,y) = e^{x^2-y^2}$ is not coercive

The domain $\{e^{x^2-y^2} \le 1\}$ has $(0,y)$ in it for any y.

$\Rightarrow$ not bounded (y can go to infinity)

Some examples of coercive functions:

- $f(x,y) = x^2+y^2$
- $f(x,y) = x^4+y^2$
- $f(x,y) = \sqrt{x^2+y^2}$
- $f(x,y,z) = x^2+y^2+z^2$ ....
- $f(x,y,z) = x^4+y^4+z^2$
- $f(x,y,z) = \sqrt{x^2+y^2+z^2}$.

Easy fact #1

$g(x,y,z)$ coercive, $g(x,y,z) \le f(x,y,z)$

$\Rightarrow f(x,y,z)$ is also coercive.

Example $f(x,y,z) = x^2+y^2+z^2+1$ is coercive

because $g(x,y,z) = x^2+y^2+z^2$ is coercive, and

$x^2+y^2+z^2 \le x^2+y^2+z^2+1$.

Easy fact #2

If $f(x,y,z)$ is coercive, then for any numbers a,b,c, $f(x-a, y-b, z-c)$ is coercive.

Example $f(x,y,z) = \sqrt{(x-1)^2+(y+2)^2+(z-3)^2}$ is coercive

Theorem Global minimum always exists for a coercive function on a closed domain.

To find global minimum, as usual, you look for the smallest value of the function at (Lagrange) critical points of the domain / boundary pieces.

Question. What about global maximum?

- If the domain is bounded;
  This is what we have done so far. From the list of values, you take the largest.
- If the domain is NOT bounded:
  Coercive functions have no global max.

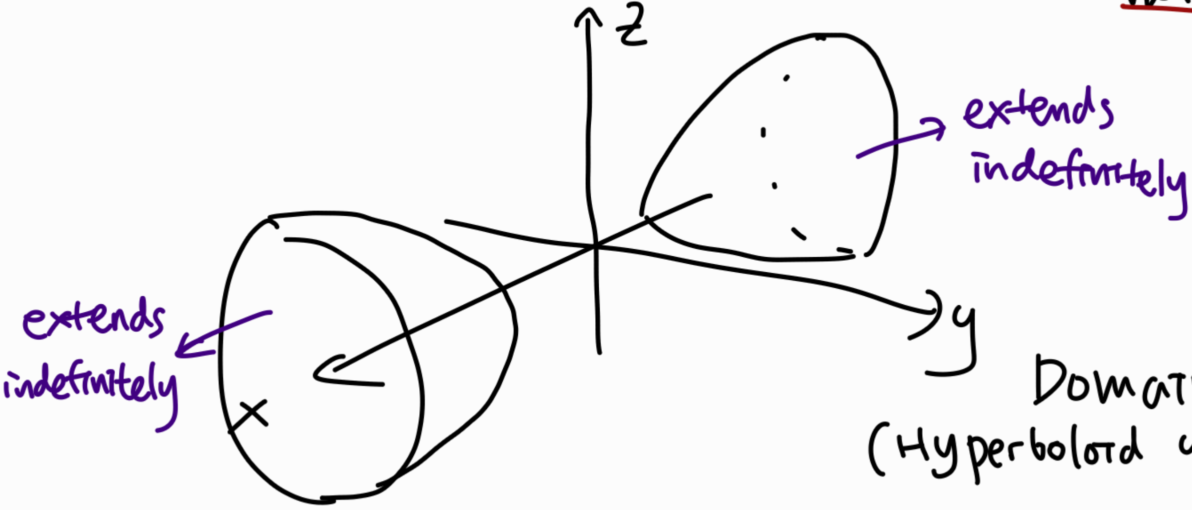
In particular, the distance between a point and any closed domain can be computed using Lagrange multipliers.

Example Find the distance between $(0,0,0)$ and the surface $x^2-y^2-z^2=1$.

This is a global minimum question in disguise:

$\boxed{\text{Find the global minimum value of } f(x,y,z)=\sqrt{x^2+y^2+z^2} \text{ on the domain } \{x^2-y^2-z^2=1\}.}$

Note that the domain $\{x^2-y^2-z^2=1\}$ is closed but not bounded



Domain
(Hyperboloid with two sheets)

But we already saw $f(x,y,z)=\sqrt{x^2+y^2+z^2}$ is coercive

$\Rightarrow$ We can just use the usual procedure (3 variables, 1 equality) for the global minimum. (Not applicable for global maximum).

Write $g(x,y,z)=x^2-y^2-z^2$.

Lagrange critical points of the original domain.

- Case A $\nabla g(x,y,z)=\langle 0,0,0\rangle$ (w/ $g(x,y,z)=1$)

  $\nabla g(x,y,z)=\langle 2x,-2y,-2z\rangle$, so it's $=\langle 0,0,0\rangle$ only if $x=y=z=0$, which does not satisfy $x^2-y^2-z^2=1$.

  $\Rightarrow$ $\boxed{\text{No points in this case.}}$

· Case B $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ (w/ $g(x,y,z)=1$)

$$\nabla f(x,y,z) = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$$

$\rightsquigarrow$ system of equations

Eq1 ... $\frac{x}{\sqrt{x^2+y^2+z^2}} = 2\lambda x$

Eq2 ... $\frac{y}{\sqrt{x^2+y^2+z^2}} = -2\lambda y$

Eq3 ... $\frac{z}{\sqrt{x^2+y^2+z^2}} = -2\lambda z$

Eq4 ... $x^2 - y^2 - z^2 = 1.$

From Eq1 → want to divide by x. → get $\boxed{2\lambda = \frac{1}{\sqrt{x^2+y^2+z^2}}}$
→ may not be able to divide by x → because $\boxed{x=0.}$

If $\boxed{2\lambda = \frac{1}{\sqrt{x^2+y^2+z^2}}}$ $\rightsquigarrow$ System of eq. becomes

Eq1 ... $\frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}}$ ✓

Eq2 ... $\frac{y}{\sqrt{x^2+y^2+z^2}} = -\frac{y}{\sqrt{x^2+y^2+z^2}}$ $\rightsquigarrow$ $y=-y$ $\rightsquigarrow$ $\boxed{y=0}$

Eq3 ... $\frac{z}{\sqrt{x^2+y^2+z^2}} = -\frac{z}{\sqrt{x^2+y^2+z^2}}$ $\rightsquigarrow$ $z=-z$ $\rightsquigarrow$ $\boxed{z=0}$

Eq4 ... $x^2-y^2-z^2=1.$

$y=0, z=0 \xRightarrow{\text{Eq4}} x^2=1 \Rightarrow \boxed{x=1}$ or $\boxed{x=-1}$

$\Rightarrow \boxed{(1,0,0)}, \boxed{(-1,0,0)}$ in this case

If $\boxed{x=0}$ ~ Eq4 is $-y^2-z^2=1$ IMPOSSIBLE

⇨ No points in this case

Types		(x,y,z)	$f(x,y,z)$
Lagrange critical points original domain	A	N/A	
	B	$(1,0,0)$	$f(1,0,0)=\boxed{1}$
		$(-1,0,0)$	$f(-1,0,0)=\boxed{1}$

⇨ Global minimum value is $\boxed{1}$ //

As the domain $\boxed{\{x^2-y^2-z^2=1\}}$ is NOT BOUNDED, the global max does not exist (if you wonder)